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The Moments for the Meyer–König and Zeller Operators

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We present an explicit expression for the moments $M_n e_r$, where $e_r(x) = x^r$ ($r \in \mathbb{N}_0$), of the Meyer–König and Zeller operators in terms of a Laplace integral. Furthermore we give the complete asymptotic expansion for $n \to \infty$. © 1995 Academic Press, Inc.

1. INTRODUCTION

The operators of Meyer–König and Zeller [7] in the slight modification of Cheney and Sharma [3]

$$\begin{aligned} (M_n f)(x) &= (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{k+n}{k} x^k f\left(\frac{k}{k+n}\right) \qquad (x \in [0, 1)) \\ (M_n f)(1) &= f(1) \end{aligned} \tag{1}$$

(also called Bernstein power series) were the object of several investigations in approximation theory. Of particular interest are the moments $M_n e_r$, where the functions e_r ($r = 0, 1, 2, \ldots$) are defined by $e_r \colon x \to x^r$.

In the case $r = 0, 1$ the moments are easily determined to be

$$M_n e_0 = e_0 \qquad \text{and} \qquad M_n e_1 = e_1. \tag{2}$$

Many authors have only dealt with estimates of $(M_n e_r)(x) - x^r$ in the important case $r = 2$ (see, e.g., the literature cited in [1]). In 1984 Alkemade [1, Theorem 2] was the first who succeeded in deriving an explicit expression for the second moment in terms of a hypergeometric series

$$(M_n e_2)(x) = x^2 + \frac{x(1-x)^2}{n+1}\, {}_2F_1(1, 2; n+2; x) \qquad (x \in [0, 1)). \tag{3}$$

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Furthermore, he proved the asymptotic expansion [1, Formula 4.6]

$$(M_n e_2)(x) - x^2 = \frac{x(1-x)^2}{n} + \frac{x(1-x)^2(2x-1)}{n^2} + O(n^{-3}) \qquad (n \to \infty) \tag{4}$$

which also occurs in [8]. Results for the case $r = 3$ can be found in [2].

The purpose of this paper is to extend the known results about the second moment in [1] to higher order moments. More precisely, we give a complete solution of the problem to find an explicit expression and the asymptotic expansion for all moments $M_n e_r$ $(r = 0, 1, 2, ...)$ of the Meyer–König and Zeller operators. Alkemade's formulae are based on a certain differential equation. His method does not seem to work for the general case $r \in \mathbb{N}$. Therefore, we use another approach to the matter.

For $r \in \mathbb{N}$ and $n \in \mathbb{N}$ we present an expression for $(M_n e_r)(x)$ in terms of a Laplace integral. Finally, we derive the complete asymptotic expansion of $(M_n e_r)(x) - x^r$ as $n \to \infty$ in the form

$$(M_n e_r)(x) \sim x^r + \sum_{k=1}^{\infty} c_k^{[r]}(x)\, n^{-k} \qquad (n \to \infty).$$

The coefficients $c_k^{[r]}(x)$ $(k = 1, 2, ...; r \in \mathbb{N})$ are calculated explicitly in terms of Stirling numbers of the first and second kind. Our results make completely transparent earlier partial results which mostly were obtained by cumbersome elementary calculations (see [8]).

2. An Integral Representation for $M_n e_r$

In view of (2) we can restrict our investigations to the case $r \geq 2$. For convenience, in the following let $r \geq 1$. Our starting point is the identity

$$\left(\frac{k}{k+n}\right)^r = 1 + \sum_{j=1}^{r} \binom{r}{j} \frac{(-n)^j}{(j-1)!} \int_0^{\infty} t^{j-1} e^{-(k+n)t}\, dt \tag{5}$$

which is easily seen to be valid for all $k = 0, 1, 2, ...$ and $n \in \mathbb{N}$. Then we get for every fixed $x \in (0, 1)$,

$$(M_n e_r)(x) = (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{k+n}{k} x^k \left(\frac{k}{k+n}\right)^r$$
$$= \frac{(1-x)^{n+1}}{n!} \sum_{k=0}^{\infty} \frac{d^n}{dx^n} x^{k+n} \left(\frac{k}{k+n}\right)^r$$

$$= \frac{(1-x)^{n+1}}{n!} \frac{d^n}{dx^n} \left[\frac{x^n}{1-x} + x^n \sum_{j=1}^{r} \binom{r}{j} \frac{(-n)^j}{(j-1)!} \right.$$

$$\left. \times \int_0^\infty t^{j-1} \sum_{k=0}^{\infty} e^{-(k+n)t} x^k \, dt \right]$$

$$= 1 + \frac{(1-x)^{n+1}}{n!} \sum_{j=1}^{r} \binom{r}{j} \frac{(-n)^j}{(j-1)!} \frac{d^n}{dx^n}$$

$$\times \int_0^\infty t^{j-1} e^{-nt} \frac{x^n}{1-xe^{-t}} \, dt$$

The interchanges of summation and differentiation are justified by the absolute and uniform convergence in every closed subinterval of $(0, 1)$. By the Leibniz rule we have

$$\frac{d^n}{dx^n} \frac{x^n}{1-xe^{-t}} = n! \sum_{i=0}^{n} \binom{n}{i} x^i \frac{e^{-it}}{(1-xe^{-t})^{i+1}}$$

$$= \frac{n!}{1-xe^{-t}} \left[1 + \frac{xe^{-t}}{1-xe^{-t}} \right]^n = \frac{n!}{(1-xe^{-t})^{n+1}}$$

and interchanging differentiation and integration yields the following

PROPOSITION. *For $r \in \mathbb{N}$ and $n \in \mathbb{N}$ the formula*

$$(M_n e_r)(x) = 1 + (1-x)^{n+1} \sum_{j=1}^{r} \binom{r}{j} \frac{(-n)^j}{(j-1)!} I(j-1, n, x) \tag{6}$$

with

$$I(j, n, x) = \int_0^\infty \frac{t^j e^{-nt}}{(1-xe^{-t})^{n+1}} \, dt \qquad (j \in \mathbb{N}_0, n \in \mathbb{N}, x \in [0, 1)) \tag{7}$$

is valid for each $x \in [0, 1)$.

Proof. We have to consider only the case $x = 0$ which follows from

$$I(j-1, n, 0) = n^{-j} \Gamma(j) \qquad (j, n \in \mathbb{N}).$$

Remark 1. Replacing t by $-\log t$, the integral $I(j, n, x)$ introduced in (7) becomes

$$I(j, n, x) = \int_0^1 (-\log t)^j \, t^{n-1} (1-xt)^{-n-1} \, dt \tag{8}$$

which will be of use presently. It may be worth noting that the proposition yields the very concise representation

$$(M_n e_r)(x) = 1 - \int_0^1 t^n \left(\frac{1-x}{1-xt}\right)^{n+1} d_t L_r(-\log t^n)$$

where L_r denotes the Laguerre polynomial

$$L_r(x) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \frac{x^j}{j!}.$$

Remark 2. In the case $r = 1$ the proposition gives the well-known equation $M_n e_1 = e_1$. This may be seen as follows. Using a linear transformation formula for the hypergeometric function ${}_2F_1(a, b; c; x)$ (see, e.g., [5, p. 89, formula (20)]) we get by (8)

$$\begin{aligned} I(0, n, x) &= n^{-1}\, {}_2F_1(n+1, n; n+1; x) = n^{-1}(1-x)^{-n}\, {}_2F_1(0, 1; n+1; x) \\ &= n^{-1}(1-x)^{-n} \end{aligned}$$

so that by (6)

$$(M_n e_1)(x) = 1 - (1-x) = x \qquad \text{for each} \quad x \in [0, 1)$$

which, of course, is also valid for $x = 1$.

Now we proceed to derive the desired expression for the moments $(M_n e_r)(x)$ by means of a Laplace integral. Replacing the variable t by $\log[x + (1-x)e^t]$ in (7) gives for every $n \in \mathbb{N}$,

$$I(j, n, x) = (1-x)^{-n} \int_0^\infty \log^j [x + (1-x) e^t]\, e^{-nt}\, dt.$$

Therefore, the proposition implies as our first main result a closed expression for $(M_n e_r)(x)$ in terms of a Laplace integral.

THEOREM 1. *For $r \in \mathbb{N}$ and $n \in \mathbb{N}$ the formula*

$$(M_n e_r)(x) = 1 + (1-x) \sum_{j=0}^{r-1} \binom{r}{j+1} \frac{(-n)^{j+1}}{j!} G_j(x, n) \tag{9}$$

is valid for every $x \in [0, 1]$, where

$$G_j(x, s) = \int_0^\infty F_j(x, t)\, e^{-st}\, dt \qquad (s > 0,\ x \in [0, 1)) \tag{10}$$

is the Laplace transform of

$$F_j(x, t) = \log^j [x + (1-x)e^t] \qquad (j \in \mathbb{N}_0, x \in [0, 1], t \geq 0) \qquad (11)$$

(as a function of t*).*

Remark 3. It could be of interest that Theorem 1 offers the possibility to define a quite natural extension of the moments $(M_n e_r)(x)$ to complex numbers n with $\mathrm{Re}(n) > 0$.

3. The Special Case $r = 2$

In the special case $r = 2$, Theorem 1 states

$$(M_n e_2)(x) = 1 + (1-x)[-2nG_0(x, n) + n^2 G_1(x, n)].$$

$G_0(x, n) = n^{-1}$ is obvious. Integration by parts and substitution in (10) as well as some known properties of hypergeometric functions yield

$$\begin{aligned} G_1(x, n) &= n^{-2}(1-x)^n \, {}_2F_1(n, n; n+1; x) \\ &= n^{-2}(1-x) \, {}_2F_1(1, 1; n+1; x) \\ &= n^{-2}(1-x)[1 + x(n+1)^{-1} \, {}_2F_1(1, 2; n+2; x)]. \end{aligned}$$

Therefore we get

$$(M_n e_2)(x) = x^2 + \frac{x(1-x)^2}{n+1} \, {}_2F_1(1, 2; n+2; x) \qquad (x \in [0, 1))$$

which is the above-mentioned result (3).

4. The Asymptotic Expansion for $M_n e_r$

In order to derive an asymptotic expansion for $M_n e_r$ it is sufficient, by (9), to study the behaviour of the Laplace integrals $G_j(x, s)$ for $s \to +\infty$. Using Watson's lemma (see, e.g., [5, p. 106f]) it is possible to give the complete asymptotic expansion of $G_j(x, s)$ as $s \to +\infty$.

Lemma 1. *Let* $F(t)$ *be defined and continuous on* $[0, \infty)$. *For some constants* $a, \delta > 0$ *let*

(a) $F(t)$ *be analytic for* $|t| \leq a + \delta$ *with* $F(t) = \sum_{k=0}^{\infty} a_k t^k$ *and*

(b) $|F(t)| < Ke^{bt}$ *for all real* $t \geq a$.

Then we have

$$\int_0^\infty F(t)\,e^{-st}\,dt \sim \sum_{k=0}^{\infty} a_k \Gamma(k+1)\,s^{-k-1} \qquad (s\to+\infty).$$

The latter formula means that, provided $s>b$,

$$\int_0^\infty F(t)\,e^{-st}\,dt = \sum_{k=0}^{p-1} a_k \Gamma(k+1)\,s^{-k-1} + o(s^{-p}) \qquad (s\to+\infty)$$

for all $p\in\mathbb{N}$. We remark that this is valid even if s is complex with $\mathrm{Re}(s)>b$ as $\mathrm{Re}(s)\to+\infty$.

Obviously, for every fixed $x\in[0,1)$ and all $j\in\mathbb{N}_0$, the functions $F_j(x,t)$ in (11) satisfy the assumptions of Lemma 1, where b may be chosen to be any arbitrary small positive constant. Therefore we have

$$F_j(x,t) = \sum_{k=0}^{\infty} a_k^{[j]}(x)\,t^k \tag{12}$$

in a neighbourhood of the origin $t=0$, and Lemma 1 implies

$$G_j(x,s) \sim \sum_{k=0}^{\infty} a_k^{[j]}(x)\,\Gamma(k+1)\,s^{-k-1} \qquad (s\to+\infty). \tag{13}$$

The coefficients $a_k^{[j]}(x)$ occurring in (12), resp. (13), will be determined in the following.

LEMMA 2. *For $k,j=0,1,2,\ldots$ there holds*

$$a_k^{[j]}(x) = \frac{j!}{k!}\sum_{i=j}^{k} S_i^j \sigma_k^i (1-x)^i \tag{14}$$

where the sum is to be read as 0 *if $k<j$.*

The quantities S_j^i and σ_j^i denote the Stirling numbers of the first, resp. second, kind defined by

$$x^{(j)} = \sum_{i=0}^{j} S_j^i x^i \qquad \text{and} \qquad x^j = \sum_{i=0}^{j} \sigma_j^i x^{(i)} \qquad (j\in\mathbb{N}_0)$$

where $x^{(j)} = x(x-1)\cdots(x-j+1)$ is the falling factorial.

Proof of Lemma 2. By the well-known power series expansions

$$\log^j(1+t) = j!\sum_{i=j}^{\infty} S_i^j \frac{t^i}{i!} \qquad (|t|<1, j\in\mathbb{N}_0)$$

and

$$(e^t-1)^i = i! \sum_{k=i}^{\infty} \sigma_k^i \frac{t^k}{k!} \qquad (t \in \mathbb{R},\, i \in \mathbb{N}_0)$$

(see, e.g., [4, p. 202]), we get using (11) that

$$\begin{aligned} F_j(x,t) &= \log^j[1+(1-x)(e^t-1)] \\ &= j! \sum_{i=j}^{\infty} S_i^j (1-x)^i \sum_{k=i}^{\infty} \sigma_k^i \frac{t^k}{k!} \\ &= j! \sum_{k=j}^{\infty} \frac{t^k}{k!} \sum_{i=j}^{k} S_i^j \sigma_k^i (1-x)^i \end{aligned}$$

for all t which are sufficiently small. This proves Lemma 2.

Combining (9), (10) with (13) and Lemma 2 we obtain our second main result.

THEOREM 2. *The complete asymptotic expansion for the moments $M_n e_r$ $(r \in \mathbb{N})$ of the Meyer–König and Zeller operators is*

$$(M_n e_r)(x) \sim x^r + \sum_{k=1}^{\infty} c_k^{[r]}(x)\, n^{-k} \qquad (n \to \infty) \tag{15}$$

for every $x \in [0, 1]$, where the coefficients are given by

$$c_k^{[r]}(x) = \sum_{j=1}^{r} \binom{r}{j} (-1)^j H(j-1, k+j-1, x) \tag{16}$$

and $H(j, m, x)$ is defined as

$$H(j, m, x) = \sum_{i=j}^{m} S_i^j \sigma_m^i (1-x)^{i+1} \qquad (0 \leqslant j \leqslant m). \tag{17}$$

Remark 4. In order that Theorem 2 also contains the trivial case $r=1$ for which $c_k^{[1]}=0$ $(k=1, 2, ...)$, the sum in (16) starts at $j=1$. In the general case $r \geqslant 2$ the sum in (16) actually runs from $j=2$ to $j=r$ because of $H(0, k, x)=0$ $(k=1, 2, ...)$.

Proof of Theorem 2. Taking into account that by Lemma 2 for all $x \in \mathbb{R}$

$$a_k^{[0]}(x) = \delta_{k0} \qquad \text{and} \qquad a_k^{[j]}(x) = 0 \qquad (k=0, ..., j-1; j \in \mathbb{N})$$

we obtain with regard to (13), (14), and (17)

$$G_0(x, s) = s^{-1}$$

for $j = 0$ and

$$\begin{aligned} G_j(x, s) &\sim \sum_{k=0}^{\infty} a_{k+j}^{[j]}(x)(k+j)!\, s^{-k-j-1} \\ &= j! \sum_{k=0}^{\infty} s^{-k-j-1} \sum_{i=j}^{k+j} S_i^j \sigma_{k+j}^i (1-x)^i \\ &= \frac{j!}{1-x} \sum_{k=0}^{\infty} s^{-k-j-1} H(j, k+j, x) \qquad (s \to +\infty). \end{aligned}$$

for $j \in \mathbb{N}$. Inserting this in Eq. (9) we get

$$\begin{aligned} (M_n e_r)(x) \sim 1 - r(1-x) + \sum_{k=0}^{\infty} n^{-k} \sum_{j=2}^{r} \binom{r}{j} \\ \times (-1)^j H(j-1, k+j-1, x) \qquad (n \to \infty). \end{aligned}$$

Taking advantage of the relationship $H(j-1, j-1, x) = (1-x)^j$ for all $j \in \mathbb{N}$, we see that

$$\begin{aligned} (M_n e_r)(x) \sim x^r + \sum_{k=1}^{\infty} n^{-k} \sum_{j=2}^{r} \binom{r}{j} \\ \times (-1)^j H(j-1, k+j-1, x) \qquad (n \to \infty) \end{aligned}$$

and, in view of Remark 4, the Proof of Theorem 2 is complete.

For practical use, however, Theorem 2 is not very suitable because the Stirling numbers occurring in (17) are not easy to handle. Therefore we close this note with the following

COROLLARY. *For every $r \in \mathbb{N}$ and $x \in (0, 1]$ we have the asymptotic relation*

$$\begin{aligned} (M_n e_r)(x) &= x^r + \frac{1}{n}\binom{r}{2}(1-x)^2 x^{r-1} \\ &+ \frac{x^{r-2}(1-x)^2}{n^2}\left[\binom{r}{2} x(2x-1) - \binom{r}{3} \right. \\ &\left. \times (1-x)(5x-1) + 3\binom{r}{4}(1-x)^2\right] \end{aligned}$$

$$+\frac{x^{r-3}(1-x)^2}{n^3}\left[\binom{r}{2}x^2(6x^2-6x+1)\right.$$

$$-2\binom{r}{3}x(1-x)(16x^2-11x+1)+\binom{r}{4}(1-x)^2$$

$$\times(61x^2-26x+1)-10\binom{r}{5}(1-x)^3(5x-1)$$

$$\left.+15\binom{r}{6}(1-x)^4\right]+O(n^{-4}) \qquad (n\to\infty).$$

Proof. By (17) we get

$$H(j-1,j,x)=x(1-x)^j\binom{j}{2}, \tag{18}$$

$$H(j-1,j+1,x)=\frac{x(1-x)^j}{24}(j+1)j(j-1)[(3j+2)x-4]$$

$$=x(1-x)^j\left[\binom{j}{2}(2x-1)+\binom{j}{3}(5x-1)+3\binom{j}{4}x\right] \tag{19}$$

and

$$H(j-1,j+2,x)$$

$$=\frac{x(1-x)^j}{48}(j+2)(j+1)j(j-1)[(j+1)(j+2)x^2-4(j+1)x+2]$$

$$=x(1-x)^j\left[\binom{j}{2}(6x^2-6x+1)+2\binom{j}{3}(16x^2-11x+1)\right.$$

$$\left.+\binom{j}{4}(61x^2-26x+1)+10\binom{j}{5}(5x^2-x)+15\binom{j}{6}x^2\right]. \tag{20}$$

The right-hand sides of (18), (19), and (20) are then substituted in formula (16), taking therein $k=1, 2, 3$. In view of the identity

$$\sum_{j=1}^{r}\binom{r}{j}(-1)^j\binom{j}{k}(1-x)^j=(-1)^k\binom{r}{k}x^{r-k}(1-x)^k \qquad (k\geqslant 1),$$

the corollary now easily follows from Theorem 2.

Finally, let us consider the special case $r=2$. In this case the corollary gives

$$(M_n e_2)(x) = x^2 + x(1-x)^2 \left\{ \frac{1}{n} + \frac{2x-1}{n^2} + \frac{6x^2-6x+1}{n^3} \right\} + O(n^{-4}) \qquad (n \to \infty)$$

which contains formula (4).

Remark 5. One of the referees pointed out that the last given asymptotic relation for $(M_n e_2)(x)$ may also be derived from (3) by expanding ${}_2F_1(1, 2; n+2; x)$.

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